A BRIEF NOTE ON THE INTERACTION OF AN ACTUATOR CASCADE WITH A SINGULARITY*

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Introduction

We have recently become concerned with making estimates of steady forces that may be exerted between moving blade rows and stationary blade rows or volutes. Our present interest is with time averaged forces for estimation of shaft loads and flow asymmetry forces rather than with transient processes. For this purpose we have adopted the well-known "actuator" model for the blade row in which the flow leaving the row or cascade is assumed to have a constant leaving angle. The disturbances external to this row such as a volute may be represented by distributions of vortex elements as was done for example by Domm and Hergt[1].

In the present case this singularity causes perturbations of the basic one-dimensional flow through the actuator cascade which lead to overall rotor forces and flow perturbations which are the subject of interest here. The problem then is one of constructing a velocity field that includes the disturbance (but adds no more) and satisfies the flow tangency condition leaving the blade row. With reference to Fig. 1 this requires

$$\frac{v}{u} = \cos\beta \tag{1}$$

at the row exit, $y = 0$, where v includes the disturbance velocity as

*Not presented at workshop.

well as added perturbations needed to satisfy Eq. (1). This is a particularly simple problem when the flow field leaving the actuator row is irrotational. In the next paragraph we consider two such cases where this assumption is valid.

The Actuator Cascade

Here we consider steady flow of constant total pressure through an actuator cascade. The flow leaving this cascade has a given direction β_v (see Fig. 1) as this is equivalent to the Kutta condition. We now consider two situations: in the first, we may imagine that there are disturbances downstream of the cascade. These disturbances may be due to the effect of downstream diffuser vanes or a volute structure for example. In the second, as a particularly simple example, we consider the effect of periodic changes in the blade leaving angle β_v on the leaving flow without any downstream disturbances.

(i) Downstream disturbances.

In the notation of Fig. 1, the trailing edge is situated on the real z axis. The flow is assumed to be irrotational so that complex variable methods may be used. Let us consider the problem of the interaction of the cascade with a single disturbance located in the upper half plane at $z = z_0$. This is denoted by

$$W_d(z-z_0) = u_d-iv_d \tag{2a}$$

where (u,v) are the velocity components in the (x,y) directions respectively. The effect of the disturbance gives rise to additional correction terms $w_i(z)$ which cannot have any singularities in the upper half plane. The

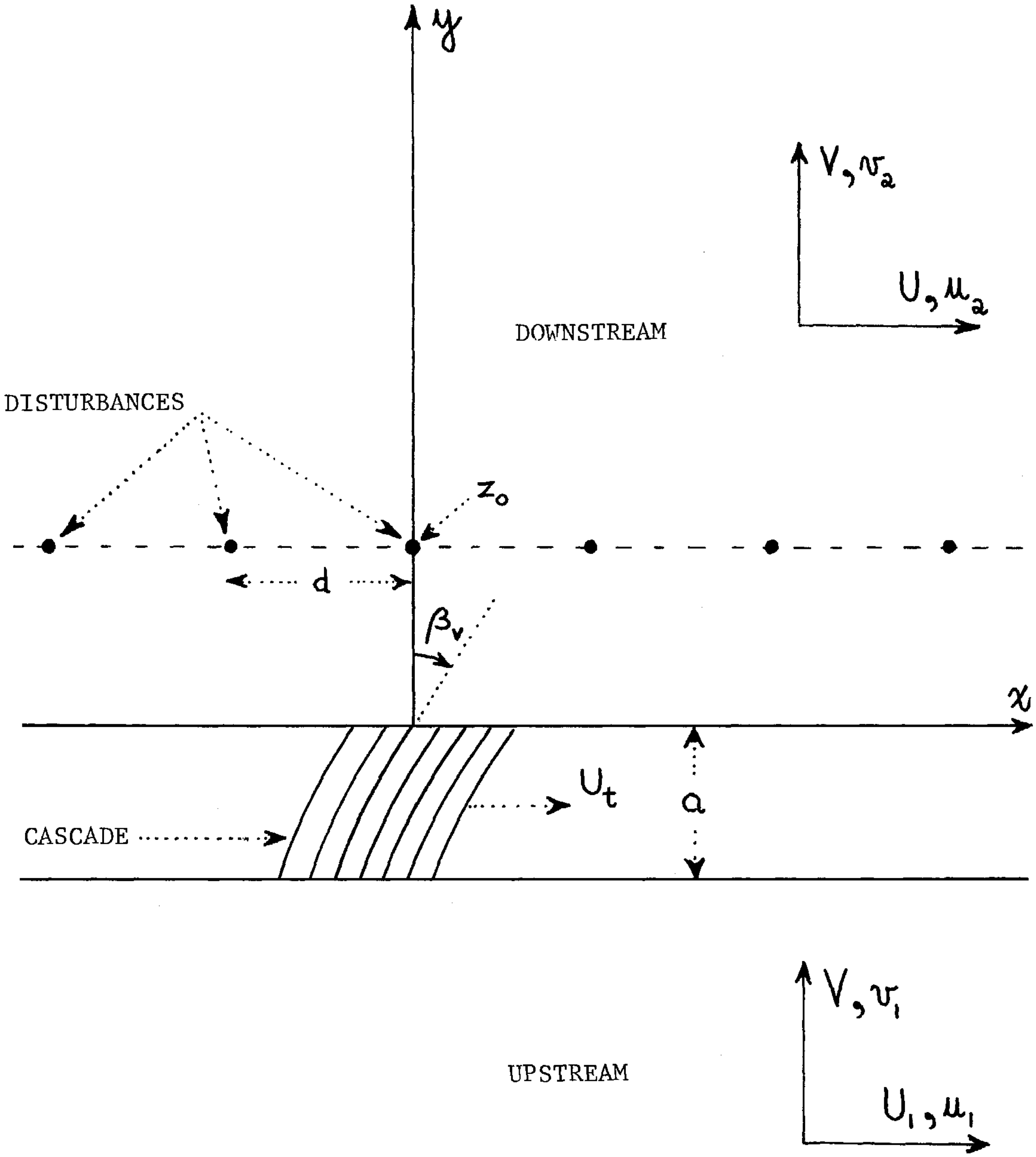

FIGURE 1. SKETCH OF CASCADE ACTUATOR ROW
($z = x + iy$)

sum

$$w_T \equiv w_d + w_i \tag{2b}$$

must satisfy the flow angle leaving condition

$$\frac{v_d+v_i+V}{u_d+u_i+U} = \cot \beta_v \tag{3}$$

where U,V are mean flow velocity components in the absence of any disturbance. Then on $y = 0$

$$v_d+v_i-(u_d+u_i)\cot \beta_v = 0 \tag{4a}$$

since $V = U \cot \beta_v$. Here (v_d,u_d) are known. The induced disturbance w_i must result in the total velocity components satisfying (4a), i.e.,

$$au_T + bv_T = 0 \tag{4b}$$

(where here $a = 1$, $b = -\tan \beta_v$). Thus w_T has to satisfy a mixed boundary condition on $y = 0$. This turns out to be neatly handled by the methods described by Cheng and Rott [2]. The induced disturbance w_i is

$$w_i = -\frac{a-ib}{a+ib}\,\bar{w}_d(z-\bar{z}_0) \tag{5}$$

which is seen merely to be an "image" of w_d in the cascade exit plane. This is easy to show; let

$$H(z) = (a-ib)w_T \quad .$$

Then $\mathrm{Re}\{H(z)\} = au + bv$ is required to be zero on the real axis. Now set

$$H(z) = (a+ib)w_d(z-z_0)-(a-ib)\bar{w}_d(z-\bar{z}_0) \quad , \tag{6}$$

so that on $y = 0, H(x,0)$ is the difference of two complex conjugate functions and is therefore purely imaginary. As an example, consider the row of vortices of strength Γ and period d seen in Fig. 1. Then

$$w_d = -\frac{i\Gamma}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{z-z_0-nd} \tag{7}$$

Then

$$w_i = -\frac{i\Gamma}{2\pi} \cdot e^{2i\beta_v} \sum_{n=-\infty}^{\infty} \frac{1}{z-\bar{z}_0-nd} ,$$

and finally w_T has the well-known sum

$$w_T = \frac{\Gamma}{2d} \left\{ \cot \frac{\pi(z-z_0)}{d} + e^{2i\beta_v} \cot \frac{\pi(z-\bar{z}_0)}{d} \right\} . \tag{8}$$

We should point out here that this image system is almost the same as that used by Domm and Hergt to set up the interference problem for a volute in the presence of a point source-vortex. The principal difference is that here the leaving angle of the cascade is fully modelled instead of being approximated in the mean.

(ii) Leaving angle variation.

In the above it was assumed that the leaving angle β_v was constant. This is not essential as the following example shows. Suppose $\beta_v = \beta_v(x)$ and that $w_d = 0$. Then Eq. (3) becomes

$$\frac{v_i+V}{u_i+U} = \cot \beta_v$$

or

$$u_i - \tan \beta_v v_i = -(U-\tan \beta_v V) .$$

Let's assume that

$$\beta_v(x) = \bar{\beta}_v + \Delta\beta(x)$$

where $\Delta\beta$ is a small change. Then w_i is proportional to $\Delta\beta$ and we have approximately

$$u_i - \tan \bar{\beta}_v v_i = \frac{V}{\cos^2 \bar{\beta}_v} \Delta\beta_v(x) \quad , \tag{9a}$$

which is of the form

$$au_i + bv_i = c(x) \quad . \tag{9b}$$

Disturbance flows of this type may again be tackled by the methods of [2]; more complete formulations are given in the book by Carrier, Krook and Pearson [3]. Again we note that Eq. (9b) is equivalent to requiring that

$$\mathrm{Re}\{(a+ib)w_i\} = c(x) \tag{9c}$$

on $y = 0$. A solution of this equation is

$$w_i = \frac{1}{a+ib} c(z) \tag{10}$$

provided $c(z)$ is chosen to have no singularities in the upper half plane. As a practical example we may imagine $c(x)$ is of the form

$$c(x) = \text{const.} \quad \cos x$$

and then it is easily seen that

$$w_i = \frac{\text{const}}{a+ib} e^{iz}$$

is the required disturbance flow since w_i vanishes for $y \to \infty$.

Moving Cascades

The above examples were for irrotational, constant-energy flows. When the cascade moves in a tangential direction (parallel to the x axis) work is done on the fluid in accordance with the Euler formula. In what follows the absolute flow is assumed to be steady and the cascade moves at speed U_t parallel to the x axis. Assume also that the Bernoulli constant upstream of the cascade, B_1, is constant everywhere. Then it follows that the leaving Bernoulli constant B_2 is given by

$$B_2 = B_1 + U_t(u_2 - u_1) - \int_1^2 \frac{\partial w}{\partial t}\, ds$$

where w is the relative velocity parallel to the blades and ds is an increment of blade arc. The relative flow is unsteady since the absolute flow is steady. Thus

$$\frac{\partial}{\partial t} = U_t \frac{\partial}{\partial x} \quad .$$

With this and continuity we find

$$B_2 = B_1 + U_t \left\{ u_2 - u_1 - \frac{\partial v_2}{\partial x} \int_1^2 \frac{dy}{\cos^2 \beta_v} \right\} \tag{11}$$

and here u_2, u_1 are tangential velocities immediately downstream of and upstream of corresponding points of the moving blade row. In general, B_2 is not constant at every point along the exit from the row and we therefore expect the leaving flow to be rotational through the relation

$$\nabla B = \underline{V} \times \underline{\omega}$$

from which we find the only component of $\underline{\omega}$ to be $\underline{k}\omega_z$ or simply

$$\omega(x,0) = \frac{U_t}{v(x,0)} \left\{ \frac{\partial}{\partial x} (u_2(x,0) - u_1(x_c,-a)) + \frac{\partial^2 v(x,0)}{\partial x^2} L \right\} \tag{11a}$$

From Fig. 1 the blade exit is at $y = 0$, the inlet is at $y = -a$ and L refers to the "length"

$$L \equiv \int_{-a}^{0} \frac{dy}{\cos^2 \beta_v(y)} \quad .$$

We see that the downstream flow is then rotational. Progress is readily made now only if we assume the disturbances to the flow field are small compared to the mean velocity components (U,V). In that case it can be assumed that ω and B are constant* on mean flow streamlines given by

$$\frac{dy}{dx} = \frac{V}{U} = \tan \alpha \quad ,$$

thus

$$\omega(x,y) = \omega(x - y \cot \alpha)$$

and

$$B(x,y) = B(x - y \cot \alpha) \quad .$$

We now separate the unknown downstream and upstream flow field into components as follows

(i) downstream $(u,v)_2 = (U,V) + (u_d,v_d) + (u_i,v_i) + (u_r,v_r)$

(ii) upstream $(u,v)_1 = (U_1,V) + (u_1,v_1)$

where (U,V), $(U_1 V)$ are mean components, (u_d,v_d) is the downstream

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*i.e., we linearize the vorticity equations.

potential disturbance, (u_i,v_i) is a downstream irrotational flow and (u_r,v_r) is a rotational (shear) flow which accounts for the vorticity

$$\omega = \frac{\partial v_r}{\partial x} - \frac{\partial u_r}{\partial y} \quad .$$

It follows that u_r,v_r are constant along lines of $x-y \cot \alpha = \text{constant}$ and that $v_r = \tan \alpha \; u_r$. The upstream disturbance (u_1,v_1) is irrotational.

In this decomposition (u_d,v_d) are given disturbances. The problem then is to find the three sets of components (u_r,v_r), (u_i,v_i) and (u_1,v_1). One relation between these is given by Eq. (11a). Two more relations are needed. One of these is given by continuity across the cascade, i.e.,

$$v_2(x,0) = v_1(x_c,-a) \tag{12}$$

(here $(x_c,-a)$ and $(x,0)$ are points corresponding to the same vane trace). The other is by the flow tangency condition at $y = 0$, i.e.,

$$u_i+u_d+u_r = (v_i+v_d+v_r)\tan \beta_v \quad . \tag{13}$$

This is apparently a complicated system of relations to solve. To sum up we have the initially unknown six velocity components (u_r,v_r), (u_i,v_i) and (u_1,v_1). U_r and v_r are related to each other though the requirement that far downstream the mean flow angle is undisturbed. Both sets (u_i,v_i) downstream and (u_1,v_1) upstream are conjugate (potential) functions so that u and v are related. There are then only three unknown functions left and we have the Eqns. (11a) (with the previous definition of ω), (12) and (13) to relate them. Thus a closed system is obtained from which solutions analogous to Eq. (8) can be found. Then in principle, complete volute actuator impeller interactions can be worked out.

We should mention that the type of problem addressed in this section is not new except in its application to singular disturbances. Earlier Ehrich [4] studied the effect of inlet wakes passing through rotor and stator blade rows with a set of equations essentially identical to the present ones. Subsequently Katz [5] carried out a similar computation using the acceleration potential instead of the velocity components but with the inclusion of losses through the blade row. Again the matching problem across the blade row is essentially the same as the present one. The interaction flow fields in these works were determined by Fourier series expansion which is a suitable procedure when only a few terms are needed to represent the disturbance.

Discussion

We have used the singularities of Eq. (8) to study the interaction between a rotating actuator impeller and a volute (assuming, irrotational volute flow). The unknown vorticity distribution on the volute is expressed in a Glauert series the coefficients of which are then determined in the usual way to make the volute surface a streamline. With these, forces can be found. This task is nearly complete; we intend subsequently to include the rotational effects described in the previous section.

Acknowledgment

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